Painlevé Analysis in Superspace

Alin A. Constandache*

Department of Physics and Astronomy

University of Rochester

February 8, 2008

Abstract

A method for carrying out the Painlevé test in superspace is proposed. The method is then applied to the one-parameter $N = 1$ supersymmetric extensions of the KdV equation.

*e-mail: alexc@pas.rochester.edu

1 Introduction

The Painlevé analysis is a simple and useful tool for testing the integrability of ordinary differential equations (ODEs) and partial differential equations (PDEs) by analyzing the singularities of the solutions. It has its origins in the works of S. Kowalewski, P. Painlevé and B. Gambier [1, 2, 3], who introduced it in the study of ODEs. The test has since been extended to PDEs by Ablowitz, Ramani and Segur [4, 5] who conjectured that any similarity reduction to an ODE, of a given PDE that is integrable by the method of inverse scattering, must have the Painlevé property. Weiss, Tabor and Carnevale [6] have proposed a version of the test that can be applied directly to PDEs, without any reduction to ODEs. This has been explicitly verified in the case of a number of integrable systems such as the Burgers' equation, the KdV equation, the Boussinesq equation, the KP equation and so on.

More recently, there has been interest in the study of supersymmetric integrable systems for a variety of reasons. In addition to the usual bosonic dynamical variables, such models also contain anti-commuting Grassmann variables. The Painlevé property of such systems have been studied only in a handful of cases and there in component formalism. The presence of fermions, in such systems, gives rise to some new properties, mainly in the recursion of the coefficient functions. The natural manifold on which a supersymmetric system is defined is a superspace, which contains Grassmann coordinates in addition to the standard bosonic coordinates. The study of the Painlevé property of a supersymmetric integrable system should naturally be carried out on this supermanifold. So far, however, a description of the Painlevé analysis in superspace is lacking [7]. In this paper, we make an attempt at generalizing the Painlevé analysis to superspace. In section **2**, we describe the general formalism of how the Painlevé analysis can be carried out in superspace. In section **3**, we apply our method to study the integrability of the $N = 1$ supersymmetric KdV system and we present a brief conclusion in section **4**.

2 Painlevé Analysis in Superspace

The Painlevé analysis, for standard bosonic systems, consists of expanding the solution as a power series

$$u = \sum_{k=0}^{\infty} u_k \rho^{k-\beta}$$

where

$$\rho = 0$$

defines the singularity manifold of the system of solutions. The PDE is said to have the Painlevé property if the solution is single valued about the movable singularity manifolds. This can happen only if the exponent β is an integer and that the coefficients u_k's are related by some recursion relation (to be determined from the dynamical equation) such that the solution is analytic near the

singularity manifold. A naive generalization of this method to superspace would consist of expanding the solution (which would be a superfield consisting of both bosonic and fermionic variables) of the dynamical equation in superspace, in the form

$$\Phi(t,x,\theta)=\sum_{k=0}^{\infty}\Phi_k\rho^{k-\beta} \tag{1}$$

where, for simplicity, we are considering a superspace with a single Grassmann variable θ. The superfield Φ, if it is fermionic, will have an expansion of the form

$$\Phi(t,x,\theta)=\psi(t,x)+\theta u(t,x) \tag{2}$$

where ψ and u are respectively the fermionic and the bosonic dynamical variables in the component form. The singularity manifold in (1) would now be a supermanifold and the coefficient functions in (2) would, in general, depend on the Grassmann coordinates of the manifold. A simple extrapolation of the results for bosonic systems would then say that the superspace equation would have the Painlevé property provided β is an integer and that we can find recursion relations between the coefficient functions such that the solution is analytic near the singularity supermanifold. A little thought, however, easily convinces one that such a generalization is doomed to fail for an obvious reason. On this superspace, there is a covariant derivative

$$D=\frac{\partial}{\partial\theta}+\theta\frac{\partial}{\partial x} \tag{3}$$

which is fermionic and is invariant under a supersymmetry transformation (translations of the superspace) satisfying

$$D^2=\frac{\partial}{\partial x} \tag{4}$$

The dynamical equations in superspace, therefore, in general, admit covariant derivatives acting on the superfield variables, in addition to the usual bosonic derivative with respect to x. As a result, it becomes impossible to obtain a recursion directly between the coefficients Φ_k in (1). We want to emphasize here that, for systems which do not explicitly involve covariant derivatives, the Painlevé analysis will go through with the expansion of the form (1). However, it would be more useful to have a general description which works in all cases.

The solution to this problem is, in fact, quite simple. Let us, for simplicity, consider a fermionic superfield $\Phi(x,\theta)=\psi(x)+\theta\,u(x)$ and an equation of motion which has the form

$$\Phi_t=P(\Phi_{(k)},\ (D\,\Phi)_{(k)}) \tag{5}$$

where $P(\Phi_{(k)},\ (D\,\Phi)_{(k)})$ is a polynomial in Φ, $(D\,\Phi)$ and their x-derivatives up to some order (we use the condensed notation $A_{(k)}\equiv\partial^k A$). Taking the covariant derivative of (5) will give us an additional equation of the form

$$(D\,\Phi)_t=Q(\Phi_{(k)},\ (D\,\Phi)_{(k)}) \tag{6}$$

where Q is another polynomial in Φ, $(D\,\Phi)$ and their x-derivatives up to some order. Let U now be a bosonic field independent of Φ and consider the system of coupled PDEs:

$$\begin{aligned} \Phi_t &= P(\Phi_{(k)},\, U_{(k)}) \\ U_t &= Q(\Phi_{(k)},\, U_{(k)}) \end{aligned} \tag{7}$$

where P and Q are the same polynomials that appear on the right hand side in (5) and (6) only this time in variables Φ and U (instead of Φ and $(D\,\Phi)$). This system has a few nice properties which make it interesting for our purpose:

(i) it does not explicitly contain the covariant derivative, hence we can perform the Painlevé test on it in the usual way;

(ii) if Φ is a solution of (5) then $(\Phi,\ (D\,\Phi))$ is a solution of (7);

(iii) if every solution of (7) is analytic in a neighborhood of the singularity manifold ρ and admits a Painlevé expansion in this neighborhood, then the same holds for every solution of (5);

Based on these observations we propose the following prescription for performing the Painlevé test in superspace:

Step 1. Obtain the coupled system (7) associated with the given PDE.

Step 2. Treating U and Φ as independent superfields, look for solutions of the form

$$U = \sum_{k=0}^{\infty} U_k \rho^{k-\alpha}, \quad \Phi = \sum_{k=0}^{\infty} \Phi_k \rho^{k-\beta}$$

where U_k are bosonic superfields, Φ_k are fermionic superfields and ρ is another bosonic superfield representing the singularity manifold.

Step 3. Find the integer values of α and β for which the system has enough resonances.

Step 4. Out of the cases found in the previous step, select the ones for which all the compatibility conditions are satisfied. These are the cases that will pass the test.

The resulting calculations can be very tedious, therefore one might want to use a simplification known in the literature as the *Kruskal ansatz*. For a bosonic system, the Kruskal ansatz consists of looking for solutions of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)\rho^{k-\beta}(x,t)$$

where $\rho(x,t) = x - \phi(t)$ with $\rho(x,t) = 0$ representing the equation of the singularity manifold. The fact that for some ϕ the singularity manifold has

this special form is guaranteed by the implicit function theorem provided that $\rho_x \neq 0$. With this in mind it is easy to see that the Kruskal ansatz generalizes to superspace as follows: for a system of the form (7) look for solutions $\Phi = \sum_{k=0}^{\infty} \Phi_k \rho^{k-\beta}$, $U = \sum_{k=0}^{\infty} U_k \rho^{k-\alpha}$ with restrictions $\rho_x = 1$, $\Phi_{k\,x} = 0$, $U_{k\,x} = 0$.

We turn our attention now to the connection that exists between the covariant version of the Painlevé analysis, as described in the previous paragraphs, and the traditional way of carrying it out in components. Let us consider the most general monomial in $\Phi, (D\Phi)$ and their x-derivatives:

$$\prod_{k=0}^{n} [\Phi_{(k)}^{p_k} \, (D\,\Phi_{(k)})^{q_k}] \tag{8}$$

where $A_{(k)} \equiv \partial^k A$, q_k are positive integers and $p_k \in \{0, 1\}$. Writing this in components we get:

$$\begin{aligned}
& \prod_{k=0}^{n} [\Phi_{(k)}^{p_k} \, (D\,\Phi_{(k)})^{q_k}] = \\
= \quad & \prod_{k=0}^{n} [(\psi_{(k)}^{p_k} + \theta\, p_k \psi_{(k)}^{p_k-1} u_{(k)})(u_{(k)}^{q_k} + \theta\, q_k u_{(k)}^{q_k-1} \psi_{(k+1)})] = \\
= \quad & \prod_{k=0}^{n} [\psi_{(k)}^{p_k} u_{(k)}^{q_k}] + \theta \sum_{k=0}^{n} (-1)^{\sigma_k} (\psi_{(0)}^{p_0} u_{(0)}^{q_0}) \dots (\psi_{(k)}^{p_k} u_{(k)}^{q_k})^{\clubsuit} \dots (\psi_{(n)}^{p_n} u_{(n)}^{q_n})
\end{aligned}$$

where $\sigma_k = \sum_{j=0}^{k-1} p_j$ and $(\psi_{(k)}^{p_k} u_{(k)}^{q_k})^{\clubsuit} = p_k u_{(k)}^{q_k+1} \psi_{(k)}^{p_k-1} + (-1)^{p_k} q_k u_{(k)}^{q_k-1} \psi_{(k)}^{p_k} \psi_{(k+1)}$. Hence the components of (8) are

$$\begin{cases} \prod_{k=0}^{n} [\psi_{(k)}^{p_k} u_{(k)}^{q_k}] \\ \sum_{k=0}^{n} (-1)^{\sigma_k} (\psi_{(0)}^{p_0} u_{(0)}^{q_0}) \dots (\psi_{(k)}^{p_k} u_{(k)}^{q_k})^{\clubsuit} \dots (\psi_{(n)}^{p_n} u_{(n)}^{q_n}) \end{cases} \tag{9}$$

If we compute the covariant derivative of (8) we obtain:

$$\begin{aligned}
& D \left(\prod_{k=0}^{n} (\Phi_{(k)}^{p_k} \, (D\,\Phi_{(k)})^{q_k} \right) = \\
= \quad & \sum_{k=0}^{n} (-1)^{\sigma_k} (\Phi_{(0)}^{p_0} (D\,\Phi_{(0)})^{q_0}) \dots D\,(\Phi_{(k)}^{p_k} (D\,\Phi_{(k)})^{q_k}) \dots (\Phi_{(n)}^{p_n} (D\,\Phi_{(n)})^{q_n}) = \\
= \quad & \sum_{k=0}^{n} (-1)^{\sigma_k} (\Phi_{(0)}^{p_0} (D\,\Phi_{(0)})^{q_0}) \dots (\Phi_{(k)}^{p_k} (D\,\Phi_{(k)})^{q_k})^{\clubsuit} \dots (\Phi_{(n)}^{p_n} (D\,\Phi_{(n)})^{q_n})
\end{aligned} \tag{10}$$

Denoting $(D\,\Phi)$ by U in (8) and (10) we obtain two polynomials which have the same structure as the components of (8). Due to the linearity of D, this property extends to arbitrary polynomials, which leads to the following result: given a supersymmetric PDE of the form (5) one can associate with it a system

of coupled PDEs (7) as previously describred. If the form of (5) in components is

$$\begin{aligned} \psi_t &= p(\psi_{(k)}, u_{(k)}) \\ u_t &= q(\psi_{(k)}, u_{(k)}) \end{aligned} \tag{11}$$

then the polynomials P and p have identical structure and so do Q and q. This means that the initial PDE passes the covariant version of the Painlevé test if and only if it passes the traditional componentwise version of the test.

Finally, we note that a similar argument can be made if the superfield we start with is a bososnic superfield.

3 Application to $N = 1$ susy-KdV

In this section we apply the superspace Painlevé analysis, as defined in section 2, to the $N = 1$ supersymmetric extensions of the KdV equation with one arbitrary parameter and we regain the known result that for only two values of the parameter ($c = 0$ and $c = 3$) the system is integrable.

The one-parameter family of supersymmetric extensions has the form:

$$\Phi_t = \Phi_{xxx} + (6-c)\,(D\Phi)\,\Phi_x + c\,\Phi\,(D\Phi)_x \tag{12}$$

therefore the coupled system we will analyze is

$$\begin{aligned} \Phi_t &= \Phi_{xxx} + (6-c)\,U\,\Phi_x + c\,\Phi\,U_x \\ U_t &= U_{xxx} + 6\,U\,U_x - c\,\Phi\,\Phi_{xx} \end{aligned} \tag{13}$$

Note that this no longer involves the covariant derivative explicitly. For simplicity we will use the Kruskal ansatz, namely we will look for expansions of the form

$$U = \textstyle\sum_{k=0}^{\infty} U_k \rho^k, \quad \Phi = \sum_{k=0}^{\infty} \Phi_k \rho^k \tag{14}$$

with the restrictions $U_{kx} = 0$, $\Phi_{kx} = 0$, $\rho_x = 1$. Plugging this back into (13) we get:

$$\begin{aligned} \sum_{k=0}^{\infty} \Phi_{kt}\,\rho^k \quad &+ \quad \sum_{k=0}^{\infty} (k-\beta)\,\Phi_k\,\rho_t\,\rho^{k-1} = \\ &= \quad \sum_{k=0}^{\infty} (k-\beta)(k-\beta-1)(k-\beta-2)\,\Phi_k\,\rho^{k-3} + \\ &+ \quad (6-c)\,\rho^{-\alpha} \sum_{k=0}^{\infty} \sum_{l=0}^{k} (l-\beta)\,U_{k-l}\,\Phi_l\,\rho^{k-1} + \end{aligned}$$

$$
\begin{aligned}
&+ \quad c\,\rho^{-\alpha} \sum_{k=0}^{\infty} \sum_{l=0}^{k} (l-\alpha)\,\Phi_{k-l}\,U_l\,\rho^{k-1} \\
\sum_{k=0}^{\infty} U_{kt}\,\rho^k \quad &+ \quad \sum_{k=0}^{\infty} (k-\alpha)\,U_k\,\rho_t\,\rho^{k-1} = \\
&= \quad \sum_{k=0}^{\infty} (k-\alpha)(k-\alpha-1)(k-\alpha-2)\,U_k\,\rho^{k-3} + \\
&+ \quad 6\,\rho^{-\alpha} \sum_{k=0}^{\infty} \sum_{l=0}^{k} (l-\alpha)\,U_{k-l}\,U_l\,\rho^{k-1} - \\
&- \quad c\,\rho^{-(2\beta-\alpha)} \sum_{k=0}^{\infty} \sum_{l=0}^{k} (l-\beta)(l-\beta-1)\,\Phi_{k-l}\Phi_l\,\rho^{k-2}
\end{aligned}
\tag{15}
$$

In order for the system to pass the test, it must have six resonances, hence we must have a polynomial of order six in k in the determinant of the recursion matrix. This is possible only if $\alpha = 2$ and $\beta = 2$.

Equating coefficients in (15) we get

$$
\begin{aligned}
\Phi_{k-3\,t} + (k-4)\,\Phi_{k-2}\,\rho_t \quad &= \quad (k-2)(k-3)(k-4)\,\Phi_k + \\
&+ \quad (6-c) \sum_{l=0}^{k} (l-2)\,U_{k-l}\,\Phi_l + \\
&+ \quad c \sum_{l=0}^{k} (l-2)\,\Phi_{k-l}\,U_l \\
U_{k-3\,t} + (k-4)\,U_{k-2}\,\rho_t \quad &= \quad (k-2)(k-3)(k-4)\,U_k + \\
&+ \quad 3\,(k-4) \sum_{l=0}^{k} U_{k-l}\,U_l - \\
&- \quad c\,(k-4) \sum_{l=0}^{k+1} l\Phi_{k+1-l}\,\Phi_l
\end{aligned}
\tag{16}
$$

In particular, for $k = 0$, this becomes:

$$
\begin{aligned}
\Phi_0\,(U_0+2) \quad &= \quad 0 \\
-3\,U_0\,(2+U_0) + c\,\Phi_0\,\Phi_1 \quad &= \quad 0
\end{aligned}
\tag{17}
$$

There are only two ways to satisfy these two equations simultaneously:

(i) $U_0 = -2$ and $\Phi_0\,\Phi_1 = 0$

(ii) $U_0 = -2$ and $c = 0$

The last of the two cases corresponds to the susy-KdVB equation and it also arises as a special subcase of the first, as we will see later. Therefore, without any loss of generality we can restrict our analysis to the first case. The recurrence then takes the form:

$$\begin{pmatrix} k\,(k^2 - 9\,k + 14 + 2\,c) & (c\,k - 12)\,\Phi_0 \\ -c(k-4)(k-1)\,\Phi_1 & (k-4)(k+1)(k-6) \end{pmatrix} \begin{pmatrix} \Phi_k \\ U_k \end{pmatrix} = \begin{pmatrix} F_k \\ G_k \end{pmatrix} \tag{18}$$

where F_k depends on $U_0, \ldots, U_{k-1}, \Phi_0, \ldots, \Phi_{k-1}$ and their derivatives, while G_k depends on $U_0, \ldots, U_{k-1}, \Phi_0, \ldots, \Phi_{k-1}, \Phi_{k+1}$ and their derivatives.

The resonances are therefore given by the roots of the polynomial

$$k\,(k-4)(k+1)(k-6)(k^2 - 9\,k + 14 + 2\,c)$$

and in order to have six of them, the quadratic factor must have two integer roots, both greater or equal to -1. This restriction leaves us with only five possible values for c:

(i) $c = 3$.

The system has resonances at $-1, 0, 4, 5, 6$ and the resonance at $k = 4$ has multiplicity 2. We checked that all the compatibility conditions for this case are satisfied and therefore it passes the test, as expected. The arbitrary functions are $\rho, \Phi_0, \Phi_4, U_4, \Phi_5, U_6$.

(ii) $c = 2$

The resonances occur at $-1, 0, 3, 4, 6$ and the resonance $k = 6$ has multiplicity 2. However, the compatibility condition for $k = 3$ is not satisfied.

(iii) $c = 0$.

This is the susy-KdVB case that we have mentioned before. The resonances are $-1, 0, 2, 4, 6, 7$. All the compatibility conditions are satisfied, thus this case also passes the test. The arbitrary functions are $\rho, \Phi_0, \Phi_2, U_4, U_6, \Phi_7$.

(iv) $c = -3$

The resonances are $-1, 0, 1, 4, 6, 8$. The compatibility condition at $k = 6$ does not hold.

(v) $c = -7$

The resonances are $-1, 0, 4, 6, 9$, with the one at $k = 0$ having multiplicity 2. However, the compatibility condition for $k = 0$ is not satisfied.

Therefore, out of the five possible cases, only the two that are known to be integrable have the Painlevé property.

4 Conclusion

We have generalised the Painlevé analysis to superspace by introducing an additional superfield such that all explicit occurences of the covariant derivative in the resulting equations of motion are eliminated. The Painlevé analysis can then be carried out for the resulting system. We have applied the method to the $N = 1$ extensions of the KdV equation and regained the known result that only two of these extensions have the Painlevé property. However, unlike earlier analysis [9], here we work manifestly in superspace. The application of our method to $N = 2$ supersymmetric systems is under investigation and will be reported in the future.

Acknowledgements

This work has been supported in part by the U.S. Deptartment of Energy Grant DE-FG 02-91ER40685.

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